

**JEE–MAIN EXAMINATION – JANUARY 2025**

(HELD ON WEDNESDAY 29<sup>TH</sup> JANUARY 2025)

TIME : 9:00 AM TO 12:00 NOON

**MATHEMATICS**

**TEST PAPER WITH SOLUTION**

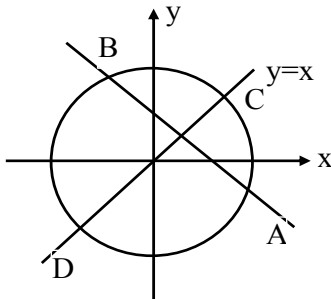
**SECTION-A**

1. Let the line  $x + y = 1$  meet the circle  $x^2 + y^2 = 4$  at the points A and B. If the line perpendicular to AB and passing through the mid point of the chord AB intersects the circle at C and D, then the area of the quadrilateral ADBC is equal to

- (1)  $3\sqrt{7}$                       (2)  $2\sqrt{14}$   
 (3)  $5\sqrt{7}$                       (4)  $\sqrt{14}$

Ans. (2)

Sol.



By solving  $x = y$  with circle

We get

$$C(\sqrt{2}, \sqrt{2})$$

$$D(-\sqrt{2}, -\sqrt{2})$$

By solving  $x + y = 1$  with circle  $x^2 + y^2 = 4$

we set

$$A\left(\frac{1+\sqrt{7}}{2}, \frac{1-\sqrt{7}}{2}\right)$$

$$\& B\left(\frac{1-\sqrt{7}}{2}, \frac{1+\sqrt{7}}{2}\right)$$

$\therefore$  Area of Quadrilateral ACBD

$$= 2 \times \text{Area of } \triangle BCD$$

$$= 2 \times \frac{1}{2} \begin{vmatrix} \sqrt{2} & \sqrt{2} & 1 \\ \frac{1-\sqrt{7}}{2} & \frac{1+\sqrt{7}}{2} & 1 \\ -\sqrt{2} & -\sqrt{2} & 1 \end{vmatrix}$$

$$= 2\sqrt{14}$$

2. Let M and m respectively be the maximum and the minimum values of

$$f(x) = \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 4x \\ \sin^2 x & 1 + \cos^2 x & 4 \sin 4x \\ \sin^2 x & \cos^2 x & 1 + 4 \sin 4x \end{vmatrix}, x \in \mathbb{R}$$

Then  $M^4 - m^4$  is equal to :

- (1) 1280                      (2) 1295  
 (3) 1040                      (4) 1215

Ans. (1)

Sol. 
$$\begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 4x \\ \sin^2 x & 1 + \cos^2 x & 4 \sin 4x \\ \sin^2 x & \cos^2 x & 1 + 4 \sin 4x \end{vmatrix}, x \in \mathbb{R}$$

$$R_2 \rightarrow R_2 - R_1, \& R_3 \rightarrow R_3 - R_1$$

$$f(x) \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 4x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expand about  $R_1$ , we get

$$f(x) = 2 + 4\sin 4x$$

$$\therefore M = \text{max value of } f(x) = 6$$

$$m = \text{min value of } f(x) = -2$$

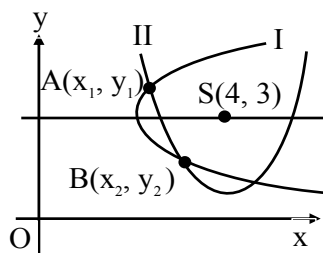
$$\therefore M^4 - m^4 = 1280$$

3. Two parabolas have the same focus (4,3) and their directrices are the x-axis and the y-axis, respectively. If these parabolas intersect at the points A and B, then  $(AB)^2$  is equal to

- (1) 192                      (2) 384  
 (3) 96                      (4) 392

Ans. (1)

Sol.



Let intersection points of these two parabolas are  $A(x_1, y_1)$  &  $B(x_2, y_2)$

$\therefore$  equation of parabola I and II are given below

$$\therefore (x-4)^2 + (y-3)^2 = x^2 \quad \dots(1)$$

$$\& (x-4)^2 + (y-3)^2 = y^2 \quad \dots(2)$$

Here  $A(x_1, y_1)$  &  $B(x_2, y_2)$  will satisfy the equation

Also from equations (1) & (2), we get  $x = y \dots(3)$

Put  $x = y$  in equation (1)

$$\text{We get } x^2 - 14x + 25 = 0$$

$$x_1 + x_2 = 14$$

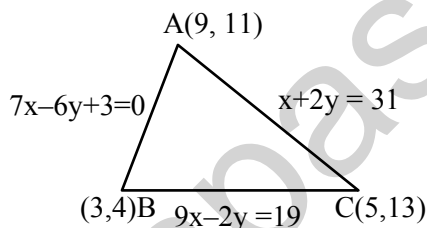
$$x_1 x_2 = 25$$

$$\begin{aligned} \therefore AB^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= 2(x_1 - x_2)^2 \\ &= 2[(x_1 + x_2)^2 - 4x_1 x_2] \\ &= 192 \end{aligned}$$

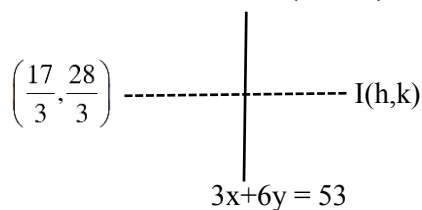
4. Let ABC be a triangle formed by the lines  $7x - 6y + 3 = 0$ ,  $x + 2y - 31 = 0$  and  $9x - 2y - 19 = 0$ , Let the point  $(h, k)$  be the image of the centroid of  $\Delta ABC$  in the line  $3x + 6y - 53 = 0$ . Then  $h^2 + k^2 + hk$  is equal to

- (1) 37 (2) 47  
(3) 40 (4) 36

Ans. (1)  
Sol.



$$\begin{aligned} \therefore \text{centroid of } \Delta ABC &= \left( \frac{9+3+5}{3}, \frac{11+4+13}{3} \right) \\ &= \left( \frac{17}{3}, \frac{28}{3} \right) \end{aligned}$$



Let image of centroid with respect to line mirror is  $(h, k)$

$$\therefore \left( \frac{k - \frac{28}{3}}{h - \frac{17}{3}} \right) \left( -\frac{1}{2} \right) = -1$$

$$\& 3 \left( \frac{h + \frac{17}{3}}{2} \right) + 6 \left( \frac{k + \frac{28}{3}}{2} \right) = 53$$

Solving (1) & (2) we get  $h = 3, k = 4$

$$\therefore h^2 + k^2 + hk = 37$$

5. Let  $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ ,  $\vec{b} = 3\hat{i} - 5\hat{j} + \hat{k}$  and  $\vec{c}$  be a vector such that  $\vec{a} \times \vec{c} = \vec{c} \times \vec{b}$  and  $(\vec{a} + \vec{c}) \cdot (\vec{b} + \vec{c}) = 168$ . Then the maximum value of  $|\vec{c}|^2$  is:
- (1) 77 (2) 462  
(3) 308 (4) 154

Ans. (3)

Sol.  $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$   
 $\vec{b} = 3\hat{i} - 5\hat{j} + \hat{k}$   
 $\vec{a} \times \vec{c} = \vec{c} \times \vec{b}$   
 $\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = 0$   
 $(\vec{a} + \vec{b}) \times \vec{c} = 0$   
 $\Rightarrow \vec{c} = \lambda(\vec{a} + \vec{b})$

$$\vec{c} = \lambda(5\hat{i} - 6\hat{j} + 4\hat{k}) \dots(1)$$

$$|\vec{c}|^2 = \lambda^2(25 + 36 + 16)$$

$$|\vec{c}|^2 = 77\lambda^2$$

$$(\vec{a} + \vec{c}) \cdot (\vec{b} + \vec{c}) = 168$$

$$\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{c} \cdot \vec{b} + |\vec{c}|^2 = 168$$

$$14 + \vec{c} \cdot (\vec{a} + \vec{b}) + 77\lambda^2 = 168$$

using equation (1)

$$\lambda |5\hat{i} - 6\hat{j} + 4\hat{k}|^2 + 77\lambda^2 = 154$$

$$77\lambda + 77\lambda^2 - 154 = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$\lambda = -2, 1$$

$\therefore$  Maximum value of  $|\vec{c}|^2$  occurs when  $\lambda = -2$

$$|\vec{c}|^2 = 77\lambda^2$$

$$= 77 \times 4$$

$$= 308$$



10. Let  $y = y(x)$  be the solution of the differential equation

$$\cos x (\log_e(\cos x))^2 dy + (\sin x - 3y \sin x \log_e(\cos x)) dx = 0,$$

$x \in \left(0, \frac{\pi}{2}\right)$ . If  $y\left(\frac{\pi}{4}\right) = \frac{-1}{\log_e 2}$ , then  $y\left(\frac{\pi}{6}\right)$  is :

(1)  $\frac{2}{\log_e(3) - \log_e(4)}$       (2)  $\frac{1}{\log_e(4) - \log_e(3)}$

(3)  $-\frac{1}{\log_e(4)}$                       (4)  $\frac{1}{\log_e(3) - \log_e(4)}$

**Ans. (4)**

**Sol.**  $\cos x (\ln(\cos x))^2 dy + (\sin x - 3y(\sin x) \ln(\cos x)) dx = 0$

$$\cos x (\ln(\cos x))^2 \frac{dy}{dx} - 3 \sin x \cdot \ln(\cos x) y = -\sin x$$

$$\frac{dy}{dx} - \frac{3 \tan x}{\ln(\cos x)} y = \frac{-\tan x}{(\ln(\cos x))^2}$$

$$\frac{dy}{dx} + \frac{3 \tan x}{\ln(\sec x)} y = \frac{-\tan x}{(\ln(\sec x))^2}$$

$$\text{I.F.} = e^{\int \frac{3 \tan x}{\ln(\sec x)} dx} = (\ln(\sec x))^3$$

$$y \times (\ln(\sec x))^3 = -\int \frac{\tan x}{(\ln(\sec x))^2} (\ln(\sec x))^3 dx$$

$$y \times (\ln(\sec x))^3 = -\frac{1}{2} (\ln(\sec x))^2 + C$$

Given :  $x = \frac{\pi}{4}$ ,  $y = -\frac{1}{\ln 2}$

$$\frac{-1}{\ln 2} \times (\ln \sqrt{2})^3 = -\frac{1}{2} \times (\ln \sqrt{2})^2 + C$$

$$\Rightarrow \frac{-1}{8 \ln 2} \times (\ln 2)^3 = -\frac{1}{2} \times \frac{1}{4} (\ln 2)^2 + C$$

$$-\frac{1}{8} (\ln 2)^2 = -\frac{1}{8} (\ln 2)^2 + C$$

$$\Rightarrow C = 0$$

$$\therefore y (\ln(\sec x))^3 = \frac{-1}{2} (\ln(\sec x))^2 + 0$$

$$y = \frac{-1}{2 \ln(\sec x)}$$

$$y = \frac{1}{2 \ln(\cos x)}$$

$$\therefore y\left(\frac{\pi}{6}\right) = \frac{1}{2 \ln\left(\cos \frac{\pi}{6}\right)}$$

$$= \frac{1}{2 \ln\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{2\left(\frac{1}{2} \ln 3 - \ln 2\right)}$$

$$= \frac{1}{\ln 3 - \ln 4}$$

**Option (4)**

11. Define a relation R on the interval  $\left[0, \frac{\pi}{2}\right]$  by  $x R y$

if and only if  $\sec^2 x - \tan^2 y = 1$ . Then R is :

- (1) an equivalence relation
- (2) both reflexive and transitive but not symmetric
- (3) both reflexive and symmetric but not transitive
- (4) reflexive but neither symmetric nor transitive

**Ans. (1)**

**Sol.**  $\sec^2 x - \tan^2 x = 1$  (on replacing y with x)

$\Rightarrow$  Reflexive

$$\sec^2 x - \tan^2 y = 1$$

$$\Rightarrow 1 + \tan^2 x + 1 - \sec^2 y = 1$$

$$\Rightarrow \sec^2 y - \tan^2 x = 1$$

$\Rightarrow$  symmetric

$$\sec^2 x - \tan^2 y = 1,$$

$$\sec^2 y - \tan^2 z = 1$$

Adding both

$$\Rightarrow \sec^2 x - \tan^2 y + \sec^2 y - \tan^2 z = 1 + 1$$

$$\sec^2 x + 1 - \tan^2 z = 2$$

$$\sec^2 x - \tan^2 z = 1$$

$\Rightarrow$  Transitive

hence equivalence relation

**Option (1)**



$\Rightarrow \lambda = 3$  and  $\mu = 1$

$\Rightarrow \vec{a} + \vec{b} = 3\hat{i} + 9\hat{j} + 4\hat{k}$

$L_3 : \vec{r} = 2\hat{i} + 8\hat{j} + 4\hat{k} + \alpha(3\hat{i} + 9\hat{j} + 4\hat{k})$

For  $\alpha = 2, \vec{r} = 8\hat{i} + 26\hat{j} + 12\hat{k}$

15. The value of  $\lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!} \right)$  is :

(1)  $\frac{4}{3}$  (2) 2

(3)  $\frac{7}{3}$  (4)  $\frac{5}{3}$

Ans. (4)

Sol.  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!}$   
 $= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 6 - 1}{(k+3)!}$   
 $= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(k+1)(k+2)(k+3) - 1}{(k+3)!}$   
 $= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(k+1)(k+2)(k+3)}{(k+3)!} - \frac{1}{(k+3)!}$   
 $= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{1}{k!} - \frac{1}{(k+3)!} \right)$   
 $= \lim_{n \rightarrow \infty} \left( \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} - \frac{1}{4!} - \frac{1}{5!} - \frac{1}{6!} - \dots - \frac{1}{(n+3)!} \right)$   
 $= \frac{1}{1} + \frac{1}{2} + \frac{1}{6} = \frac{10}{6} = \frac{5}{3}$

16. The integral  $80 \int_0^{\frac{\pi}{4}} \left( \frac{\sin \theta + \cos \theta}{9 + 16 \sin 2\theta} \right) d\theta$  is equal to :

(1)  $3 \log_e 4$  (2)  $6 \log_e 4$

(3)  $4 \log_e 3$  (4)  $2 \log_e 3$

Ans. (3)

Sol.  $I = 80 \int_0^{\frac{\pi}{4}} \left( \frac{\sin \theta + \cos \theta}{9 + 16(2 \sin \theta \cos \theta)} \right) d\theta$   
 $= 80 \int_0^{\frac{\pi}{4}} \frac{\sin \theta + \cos \theta}{9 - 16(1 - 2 \sin \theta \cos \theta - 1)} d\theta$

$= 80 \int_0^{\frac{\pi}{4}} \frac{\sin \theta + \cos \theta}{9 + 16 - 16(\sin \theta - \cos \theta)^2} d\theta$

Let  $\sin \theta - \cos \theta = t$

$(\cos \theta + \sin \theta) d\theta = dt$

$= 80 \int_{-1}^0 \frac{dt}{25 - 16t^2}$

$= \frac{80}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$

$= \frac{5}{2} \ln \left| \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right| \Big|_{-1}^0$

$= 2 \ln(1) + 4 \ln 3$

$= 4 \ln 3$

17. Let  $L_1 : \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{2}$  and

$L_2 : \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z}{1}$  be two lines.

Let  $L_3$  be a line passing through the point  $(\alpha, \beta, \gamma)$  and be perpendicular to both  $L_1$  and  $L_2$ . If  $L_3$  intersects  $L_1$ , then  $|5\alpha - 11\beta - 8\gamma|$  equals :

(1) 18

(2) 16

(3) 25

(4) 20

Ans. (3)

Sol. DR's of  $L_3 = \vec{m} \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{vmatrix}$

$= -5\hat{i} - 3\hat{j} + \hat{k}$

$L_3 : \frac{x-\alpha}{-5} = \frac{y-\beta}{-3} = \frac{z-\gamma}{1} = \lambda$

$A(\alpha - 5\lambda, \beta - 3\lambda, \gamma + \lambda)$

$L_1 : \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{2} = k$

$B(k+1, -k+2, 2k+1)$

Now

$\alpha - 5\lambda = k+1 \Rightarrow \alpha = 5\lambda + k+1$

$\beta - 3\lambda = -k+2 \Rightarrow \beta = 3\lambda - k+2$

$\gamma + \lambda = 2k+1 \Rightarrow \gamma = -\lambda + 2k+1$

$|5\alpha - 11\beta - 8\gamma| = |-25|$

$= 25$



**SECTION-B**

21. Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a twice differentiable

function. If for some  $a \neq 0$ ,  $\int_0^1 f(\lambda x) d\lambda = af(x)$ ,

$f(1) = 1$  and  $f(16) = \frac{1}{8}$ , then  $16 - f'\left(\frac{1}{16}\right)$  is equal

to \_\_\_\_\_,

**Ans. (112)**

**Sol.**  $\int_0^1 f(\lambda x) d\lambda = af(x)$

$$\lambda x = t$$

$$d\lambda = \frac{1}{x} dt$$

$$\frac{1}{x} \int_0^x f(t) dt = af(x)$$

$$\int_0^x f(t) dt = axf(x)$$

$$f(x) = a(x f'(x) + f(x))$$

$$(1 - a)f(x) = a.x f'(x)$$

$$\frac{f'(x)}{f(x)} = \frac{(1-a)}{a} \frac{1}{x}$$

$$\ln f(x) = \frac{1-a}{a} \ln x + c$$

$$x = 1, f(1) = 1 \Rightarrow c = 0$$

$$x = 16, f(16) = \frac{1}{8}$$

$$\frac{1}{8} = (16)^{\frac{1-a}{a}} \Rightarrow -3 = \frac{4-4a}{a} \Rightarrow a = 4$$

$$f(x) = x^{-\frac{3}{4}}$$

$$f'(x) = -\frac{3}{4} x^{-\frac{7}{4}}$$

$$\therefore 16 - f'\left(\frac{1}{16}\right)$$

$$= 16 - \left(-\frac{3}{4}(2^{-4})^{-7/4}\right)$$

$$= 16 + 96 = 112$$

22. Let  $S = \left\{ m \in \mathbb{Z} : A^{m^2} + A^m = 3I - A^{-6} \right\}$ , where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}. \text{ Then } n(S) \text{ is equal to } \underline{\hspace{2cm}}.$$

**Ans. (2)**

**Sol.**  $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}, A^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}, A^4 = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$$

and so on

$$A^6 = \begin{bmatrix} 7 & -6 \\ 6 & -5 \end{bmatrix}$$

$$A^m = \begin{bmatrix} m+1 & -m \\ m & -m+1 \end{bmatrix},$$

$$A^{m^2} = \begin{bmatrix} m^2+1 & -m^2 \\ m^2 & -(m^2-1) \end{bmatrix}$$

$$A^{m^2} + A^m = 3I - A^{-6}$$

$$\begin{bmatrix} m^2+1 & -m^2 \\ m^2 & -(m^2-1) \end{bmatrix} + \begin{bmatrix} m+1 & -m \\ m & -m+1 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 6 \\ -6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -6 \\ 6 & -4 \end{bmatrix}$$

$$= m^2 + 1 + m + 1 = 8$$

$$= m^2 + m - 6 = 0 \Rightarrow m = -3, 2$$

$$n(S) = 2$$

23. Let  $[t]$  be the greatest integer less than or equal to  $t$ .

Then the least value of  $p \in \mathbb{N}$  for which

$$\lim_{x \rightarrow 0^+} \left( x \left( \left[ \frac{1}{x} \right] + \left[ \frac{2}{x} \right] + \dots + \left[ \frac{p}{x} \right] \right) - x^2 \left( \left[ \frac{1}{x^2} \right] + \left[ \frac{2^2}{x^2} \right] + \dots + \left[ \frac{9^2}{x^2} \right] \right) \right) \geq 1$$

is equal to \_\_\_\_\_.

**Ans. (24)**

**Sol.**  $\lim_{x \rightarrow 0^+} \left( x \left( \left[ \frac{1}{x} \right] + \left[ \frac{2}{x} \right] + \dots + \left[ \frac{p}{x} \right] \right) - x^2 \left( \left[ \frac{1}{x^2} \right] + \left[ \frac{2^2}{x^2} \right] + \left[ \frac{9^2}{x^2} \right] \right) \right) \geq 1$

$$(1 + 2 + \dots + p) - (1^2 + 2^2 + \dots + 9^2) \geq 1$$

$$\frac{p(p+1)}{2} - \frac{9 \cdot 10 \cdot 19}{6} \geq 1$$

$$p(p+1) \geq 572$$

Least natural value of p is 24

- 24.** The number of 6-letter words, with or without meaning, that can be formed using the letters of the word MATHS such that any letter that appears in the word must appear at least twice, is 4 \_\_\_\_.

**Ans. (1405)**

- Sol.** (i) Single letter is used, then no. of words = 5  
 (ii) Two distinct letters are used, then no. of words

$${}^5C_2 \times \left( \frac{6!}{2!4!} \times 2 + \frac{6!}{3!3!} \right) = 10(30 + 20) = 500$$

- (iii) Three distinct letters are used, then no. of words

$${}^5C_3 \times \frac{6!}{2!2!2!} = 900$$

Total no. of words = 1405

- 25.** Let  $S = \{x : \cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x+1)\}$ .

Then  $\sum_{x \in S} (2x-1)^2$  is equal to \_\_\_\_.

**Ans. (5)**

**Sol.**  $\cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x+1)$

$$2\cos^{-1}x - \sin^{-1}(2x+1) = \frac{3\pi}{2}$$

$$2\alpha - \beta = \frac{3\pi}{2} \text{ where } \cos^{-1}x = \alpha, \sin^{-1}(2x+1) = \beta$$

$$2\alpha = \frac{3\pi}{2} + \beta$$

$$\cos 2\alpha = \sin \beta$$

$$2\cos^2 \alpha - 1 = \sin \beta$$

$$2x^2 - 1 = 2x + 1$$

$$x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 - \sqrt{5}}{2}, \left\{ x = \frac{1 + \sqrt{5}}{2} \text{ rejected} \right\}$$

$$\therefore 4x^2 - 4x = 4$$

$$(2x - 1)^2 = 5$$