

MATHEMATICS

1. Find number of common terms in the two given series

4, 9, 14, 19, up to 25 terms and

3, 9, 15, 21 up to 37 terms

- (1) 4 (2) 7 (3) 5 (4) 3

Ans. (1)

Sol. 4, 9, 14, 19, $124 \rightarrow d_1 = 5$

3, 9, 15, 21 $219 \rightarrow d_2 = 6$

1st common term = 9 and common difference of common terms = 30

Common terms are 9, 39, 69, 99

4 common terms

2. Let $8 = 3 + \frac{3+p}{4} + \frac{3+2p}{4^2} + \dots \infty$, then p is

- (1) 9 (2) $\frac{5}{4}$ (3) 3 (4) 1

Ans. (1)

Sol. $8 = 3 + \frac{3+p}{4} + \frac{3+2p}{4^2} + \dots$ (i)

multiply both sides by $\frac{1}{4}$, we get

$$2 = \frac{3}{4} + \frac{3+p}{4^2} + \dots$$
 (ii)

Equation (i) – equation (ii)

$$\Rightarrow 6 = 3 + \frac{p}{4} + \frac{p}{4^2} + \dots$$

$$\Rightarrow 3 = \frac{p}{4\left(1 - \frac{1}{4}\right)} \Rightarrow p = 9$$

3. For $\frac{x^2}{25} + \frac{y^2}{16} = 1$, find the length of chord whose mid point is $P\left(1, \frac{2}{5}\right)$

- (1) $\frac{\sqrt{1681}}{5}$ (2) $\frac{\sqrt{1481}}{5}$ (3) $\frac{\sqrt{1781}}{5}$ (4) $\frac{\sqrt{1691}}{5}$

Ans. (4)

Sol. By $T = S_1$

$$\Rightarrow \frac{x}{25} + \frac{y}{16} = \frac{1}{25} + \frac{4}{25} \cdot \frac{1}{16}$$

$$\Rightarrow \frac{x}{25} + \frac{y}{40} = \frac{4+1}{100}$$

$$\Rightarrow \frac{x}{25} + \frac{y}{40} = \frac{1}{20}$$

$$\Rightarrow 8x + 5y = 10$$

$$\Rightarrow \frac{x^2}{25} + \left(\frac{10-8x}{5} \right)^2 \cdot \frac{1}{16} = 1$$

$$\Rightarrow \frac{x^2}{25} + \frac{4}{25} \left(\frac{5-4x}{16} \right)^2 = 1$$

$$\Rightarrow x^2 + \frac{(5-4x)^2}{4} = 25$$

$$\Rightarrow 4x^2 + (5-4x)^2 = 100$$

$$\Rightarrow 20x^2 - 8x - 15 = 0$$

$$x_1 + x_2 = 2$$

$$x_1 x_2 = \frac{-15}{4}$$

$$\begin{aligned} \text{length of chord} &= |x_1 - x_2| \sqrt{1+m^2} \\ &= \frac{\sqrt{1691}}{5} \end{aligned}$$

4. If $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, then find $f'(10)$.

Ans. (202)

Sol. $f'(x) = 3x^2 + 2x f'(1) + f''(2)$

$$f''(x) = 6x + 2f'(1)$$

$$f'''(3) = 6$$

$$f'(1) = -5$$

$$f''(2) = 2$$

$$\Rightarrow f'(10) = 300 + 20(-5) + 2$$

$$= 202$$

5. Let $\int_0^1 \frac{dx}{\sqrt{x+3} + \sqrt{x+1}} = A + B\sqrt{2} + C\sqrt{3}$ then the value of $2A + 3B + C$ is

$$(1) 3$$

$$(2) 4$$

$$(3) 5$$

$$(4) 6$$

Ans. (1)

Sol. On rationalising

$$\begin{aligned}& \int_0^1 \frac{(\sqrt{x+3} - \sqrt{x+1})}{2} dx \\&= \frac{2}{3 \cdot 2} \left\{ (x+3)^{3/2} - (x+1)^{3/2} \right\}_0^1 \\&= \frac{1}{3} \{ 8 - 3\sqrt{3} - (2\sqrt{2} - 1) \} \\&= \frac{1}{3} \{ 9 - 3\sqrt{3} - 2\sqrt{2} \} \\&= \left(3 - \sqrt{3} - \frac{2\sqrt{2}}{3} \right) : A = 3, B = -\frac{2}{3}, C = -1 \\&\therefore 2A + 3B + C = 6 - 2 - 1 = 3\end{aligned}$$

6. If $|z - i| = |z - 1| = |z + i|$, $z \in \mathbb{C}$, then the numbers of z satisfying the equation are
(1) 0 (2) 1 (3) 2 (4) 4

Ans. (2)

Sol. z is equidistant from 1, i , & $-i$
only $z = 0$ is possible
 \therefore number of z equal to 1

7. If sum of coefficients in $(1 - 3x + 10x^2)^n$ and $(1 + x^2)^n$ is A and B respectively then
(1) $A^3 = B$ (2) $A = B^3$ (3) $A = 2B$ (4) $A = B$

Ans. (2)

Sol. $A = 8^n$ $B = 2^n$
(B) $\therefore A = B^3$

8. Let a_1, a_2, \dots, a_{10} are 10 observations such that $\sum_{i=1}^{10} a_i = 50$ and $\sum_{i \neq j}^{10} a_i \cdot a_j = 1100$, then their standard deviation will be

- (1) $\sqrt{5}$ (2) $\sqrt{30}$ (3) $\sqrt{15}$ (4) $\sqrt{10}$

Ans. (1)

Sol. $(a_1 + a_2 + \dots + a_{10})^2 = 50^2$
 $\Rightarrow \sum a_i^2 + 2 \sum_{i \neq j} a_i a_j = 2500$
 $\Rightarrow \sum a_i^2 = 300$
 $\sigma^2 = \frac{\sum a_i^2}{10} - \left(\frac{\sum a_i}{10} \right)^2$
 $\Rightarrow \sigma^2 = 5 \Rightarrow \text{S.D} = \sqrt{5}$

9. If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

Statement-1 : $f(-x)$ is inverse of $f(x)$

Statement-2 : $f(x+y) = f(x)f(y)$

(1) Both are true

(2) Both are false

(3) Only statement 1 is true

(4) Only statement 2 is true

Ans. (1)

Sol. $f(x)f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= f(x+y)$$

$$\therefore f(x)f(-x) = f(0)$$

$$= I$$

10. If $a = \lim_{x \rightarrow 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$ find $a \cdot b^3$

(1) 16 (2) 32 (3) -16 (4) 48

Ans. (2)

Sol. $a = \lim_{x \rightarrow 0} \frac{\sqrt{1+x^4} - 1}{x^4 \left[\sqrt{1+\sqrt{1+x^4}} + \sqrt{2} \right]}$

$$= \lim_{x \rightarrow 0} \frac{x^4}{x^4 \left[\sqrt{1+\sqrt{1+x^4}} + \sqrt{2} \right] \left[\sqrt{1+x^4} + 1 \right]}$$

$$= \frac{1}{2\sqrt{2} \times 2} = \frac{1}{4\sqrt{2}}$$

$$b = \lim_{x \rightarrow 0} \frac{\sin^2 x}{(1 - \cos x)} \left(\sqrt{2} + \sqrt{1+\cos x} \right)$$

$$= 2 \times (\sqrt{2} + \sqrt{2}) = 4\sqrt{2}$$

$$\therefore ab^3 = (4\sqrt{2})^2 = 32$$

11. If the minimum distance of centre of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$ from any point on the parabola $y^2 = 4x$ is d , find d^2

Ans. (20)

Sol. Normal to parabola is $y = mx - 2m - m^3$

$$\text{centre } (2, 8) \rightarrow 8 = 2m - 2m - m^3$$

$$\Rightarrow m = -2$$

$$\therefore p \text{ is } (m^2, -2m) = (4, 4)$$

$$\Rightarrow d^2 = 4 + 16 = 20$$

12. If $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 3(\hat{i} - \hat{j} + \hat{k})$, $\vec{a} \times \vec{c} = \vec{b}$ & $\vec{a} \cdot \vec{c} = 3$ find $\vec{a} \cdot (\vec{c} \times \vec{b} - \vec{b} \times \vec{c})$

(1) 24

(2) -24

(3) 18

(4) 15

Ans. (1)

Sol. $[\vec{a} \vec{c} \vec{b}] = (\vec{a} \times \vec{c}) \cdot \vec{b} = |\vec{b}|^2 = 27$

$$\therefore \text{we need} = 27 - 0 - 3 = 24$$

13. Consider the line $L : 4x + 5y = 20$. Let two other lines are L_1 and L_2 which trisect the line L and pass through origin, then tangent of angle between lines L_1 and L_2 is

(1) $\frac{20}{41}$

(2) $\frac{30}{41}$

(3) $\frac{40}{41}$

(4) $\frac{10}{41}$

Ans. (2)

Sol. Let line L intersect the lines L_1 and L_2 at P and Q

$$P\left(\frac{10}{3}, \frac{4}{3}\right), Q\left(\frac{5}{3}, \frac{8}{3}\right)$$

$$\therefore m_{OA} = \frac{2}{5}$$

$$m_{OQ} = \frac{8}{5}$$

$$\tan \theta = \left| \frac{\frac{8}{5} - \frac{2}{5}}{1 + \frac{16}{25}} \right|$$

$$= \left(\frac{6}{5} \times \frac{25}{41} \right)$$

$$= \frac{30}{41}$$

14. If ${}^{n-1}C_r = (k^2 - 8) {}^nC_{r+1}$, then the range of 'k' is

- (1) $k \in (2\sqrt{2}, 3]$ (2) $k \in (2\sqrt{2}, 3)$ (3) $k \in [2, 3)$ (4) $k \in (2\sqrt{2}, 8)$

Ans. (1)

Sol. ${}^{n-1}C_r = (k^2 - 8) \frac{n}{r+1} \cdot {}^{n-1}C_r$

$$\Rightarrow k^2 - 8 = \frac{r+1}{n}$$

here $r \in [0, n-1]$

$$\Rightarrow r+1 \in [1, n]$$

$$\Rightarrow k^2 - 8 \in \left[\frac{1}{n}, 1\right]$$

$$\Rightarrow k^2 \in \left[8 + \frac{1}{n}, 9\right]$$

$$\Rightarrow k \in (2\sqrt{2}, 3]$$

15. If $\alpha x + \beta y + 9\ln|2x + 3y - 8\lambda| = x + C$ is the solution of $(2x + 3y - 2)dx + (4x + 6y - 7)dy = 0$, then $\alpha + \beta + \gamma =$

- (1) 18 (2) 19 (3) 20 (4) 21

Ans. (1)

Sol. Let $2x + 3y = t$

$$\Rightarrow 2 + 3 \frac{dy}{dx} = \frac{dt}{dx}$$

$$\text{Now } (t-2) + (2t-7) \left(\frac{dt}{dx} - 2 \right) \times \frac{1}{3} = 0$$

$$\Rightarrow -\frac{(3t-6)}{2t-7} = \frac{dt}{dx} - 2$$

$$\Rightarrow \frac{dt}{dx} = \frac{t-8}{2t-7}$$

$$\Rightarrow \int \frac{2t-7}{t-8} dt = \int dx$$

$$\Rightarrow \int 2 + \frac{9}{t-8} dt = \int dx$$

$$\Rightarrow 2t + 9\ln|t-8| = x + C$$

$$\Rightarrow 2(2x + 3y) + 9\ln|2x + 3y - 8| = x + C$$

$$\alpha = 4, \beta = 6, \gamma = 8$$

16. $f : N - \{1\} \rightarrow N$ and $f(n)$ = highest prime factor of 'n', then f is

- (1) one-one, onto (2) many-one, onto
(3) many-one, into (4) one-one, into

Ans. (3)

Sol. '4' is not image of any element \Rightarrow into

$$f(10) = 5 = f(15) \Rightarrow \text{many-one}$$

17. If $P(X)$ represent the probability of getting a '6' in the X^{th} roll of a die for the first time. Also

$$a = P(X = 3)$$

$$b = P(X \geq 3)$$

$$c = P\left(\frac{X \geq 6}{X > 3}\right), \text{ then } \frac{b+c}{a} = ?$$

Ans. (12)

$$\text{Sol. } P(X = 3) = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = a$$

$$P(X \geq 3) = \left(\frac{5}{6}\right)^2 = b$$

$$P\left(\frac{X \geq 6}{X > 3}\right) = \left(\frac{5}{6}\right)^2 = c$$

$$\therefore \frac{b+c}{a} = \frac{2\left(\frac{5}{6}\right)^2}{\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}} = 12$$

18. If the angle between two vectors $\vec{a} = \alpha\hat{i} - 4\hat{j} - \hat{k}$ and $\vec{b} = \alpha\hat{i} + \alpha\hat{j} + 4\hat{k}$ is acute then find least positive integral value of α .

- (1) 4 (2) 5 (3) 6 (4) 7

Ans. (2)

$$\text{Sol. } \vec{a} \cdot \vec{b} > 0$$

$$\Rightarrow \alpha^2 - 4\alpha - 4 > 0$$

$$\alpha < (2 - 2\sqrt{2}) \text{ or } \alpha > (2 + 2\sqrt{2})$$

19. If $S = \{1, 2, \dots, 10\}$ and $M = P(S)$,

If ARB such that $A \cap B \neq \phi$ where $A \in M, B \in M$

Then

- (1) R is reflexive and symmetric (2) Only symmetric
(3) Only reflexive (4) Symmetric and transitive

Ans. (2)

Sol. $\phi \cap \phi = \phi \Rightarrow (\phi, \phi) \notin R \Rightarrow$ not reflexive.

If $A \cap B \neq \phi \Rightarrow B \cap A \neq \phi \Rightarrow$ Symmetric

If $A \cap B \neq \phi$ and $B \cap C \neq \phi \Rightarrow A \cap C \neq \phi$

for example $A = \{1, 2\}$

$B = \{2, 3\}$

$C = \{3, 4\}$

20. If four points $(0, 0), (1, 0), (0, 1), (2k, 3k)$ are concyclic, then k is

- (1) $\frac{4}{13}$ (2) $\frac{5}{13}$ (3) $\frac{7}{13}$ (4) $\frac{9}{13}$

Ans. (2)

Sol. Equation of circle is

$$x(x-1) + y(y-1) = 0$$

$$x^2 + y^2 - x - y = 0$$

$$B(2k, 3k)$$

$$\Rightarrow 4k^2 + 9k^2 - 2k - 3k = 0$$

$$\Rightarrow 13k^2 = 5k$$

$$\Rightarrow k = 0, \frac{5}{13}$$

$$\therefore k = \frac{5}{13}$$

21. If $f(x)$ is differentiable function satisfying $f(x) - f(y) \geq \log \frac{x}{y} + x - y$, then find $\sum_{N=1}^{20} f'\left(\frac{1}{N^2}\right)$

Ans. (2890)

Sol. Let $x > y$

$$\lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y} \geq \frac{\log x - \log y}{x - y} + 1$$

$$f'(x^-) \geq \frac{1}{x} + 1$$

$\Rightarrow f'(x^-) = f'(x^+)$ as $f(x)$ is differentiable function

$$f'(x) = \frac{1}{x} + 1$$

$$f'\left(\frac{1}{N^2}\right) = N^2 + 1$$

$$\sum_{N=1}^{20} f'\left(\frac{1}{N^2}\right) = \sum (N^2 + 1) = \frac{20 \times 21 \times 41}{6} + 20 = 2890$$

Let $x < y$

$$\frac{f(x) - f(y)}{x - y} \leq \frac{\log x - \log y}{x - y} + 1$$

$$f'(x^+) \leq \frac{1}{x} + 1$$

22. Let $\frac{dx}{dt} + ax = 0$ and $\frac{dy}{dt} + by = 0$ where $y(0) = 1$, $x(0) = 2$, and $x(t) = y(t)$, then t is

- (1) $\frac{\ln 3}{a-b}$ (2) $\frac{\ln 2}{b-a}$ (3) $\frac{\ln 2}{a-b}$ (4) $\frac{\ln 3}{b-a}$

Ans. (3)

Sol. $\frac{dx}{dt} + ax = 0$

$$\Rightarrow \ln x = -at + c$$

$$x(0) = 2 \Rightarrow c = \ln 2$$

$$\therefore x = 2e^{-at}$$

$$\frac{dy}{dt} + by = 0 \Rightarrow y = e^{-bt}$$

$$x(t) = y(t)$$

$$2e^{-at} = e^{-bt}$$

$$\Rightarrow t = \frac{\ln 2}{a-b}$$

23. If $H(a, b)$ is the orthocentre of ΔABC where $A(1, 2)$, $B(2, 3)$ & $C(3, 1)$, then find $\frac{36I_1}{I_2}$ if

$$I_1 = \int_a^b x \sin(4x - x^2) dx \text{ and } I_2 = \int_a^b \sin(4x - x^2) dx$$

Ans. (72)

Sol. ΔABC is isosceles

$\Rightarrow H$ lies on angle bisector passing through $(3, 1)$ which is $x + y = 4$

$$\therefore a + b = 4$$

Now apply $(a + b - x)$ in I_1

$$2I_1 = \int_a^b 4 \sin(4x - x^2) dx$$

$$\Rightarrow 2I_1 = 4I_2$$

$$\Rightarrow \frac{I_1}{I_2} = 2$$

$$\therefore \frac{36I_1}{I_2} = 72$$

24. $f(x) = \begin{cases} \frac{\sin(x-3)}{2^{x-[x]}}, & x > 3 \\ \frac{a(x^2 - 7x + 12)}{b|x^2 - 7x + 12|}, & x < 3 \\ b, & x = 3 \end{cases}$. Find number of ordered pairs (a, b) so that f(x) is continuous

at $x = 3$

Ans. (1)

Sol. LHL = RHL = $f(3)$

$$-\frac{a}{b} = 2^1 = b$$

$$\Rightarrow b = 2 \text{ and } a = -4$$

$$\Rightarrow (a, b) = (-4, 2)$$

25. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$, $B = [B_1 \ B_2 \ B_3]$ where B_1, B_2, B_3 are column matrices such that

$$AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AB_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

α = sum of diagonal elements of B

$\beta = |B|$, then find $|\alpha^3 + \beta^3|$

Ans. (1.125)

Sol. $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \\ 1 & -2 & 0 \end{bmatrix}$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 2 \end{bmatrix}, B_3 = \begin{bmatrix} 2 \\ -\frac{5}{2} \\ -1 \end{bmatrix}$$

$$\text{Tr}(B) = -\frac{1}{2}$$

$$|B| = -1$$

$$\therefore a = -\frac{1}{2}, b = -1$$

$$|\alpha^3 + \beta^3| = \frac{9}{8} = 1.125$$

26. If $\cos(2x) - a \sin x = 2a - 7$ has a solution for $a \in [p, q]$ and $r = \tan 9^\circ + \tan 63^\circ + \tan 81^\circ + \tan 27^\circ$, then p.q. r = ?

- (1) $40\sqrt{5}$ (2) $32\sqrt{5}$ (3) $30\sqrt{5}$ (4) $48\sqrt{5}$

Ans. (4)

Sol. $2(\sin^2 x - 4) + a(\sin x + 2) = 0$

$$2(\sin x - 2) + a = 0$$

$$\Rightarrow a = 4 - 2 \sin x$$

$$a \in [2, 6]$$

$$\text{Also, } r = \left(\tan 9^\circ + \frac{1}{\tan 9^\circ} \right) + \left(\tan 27^\circ + \frac{1}{\tan 27^\circ} \right)$$

$$= \frac{2}{\sin 18^\circ} + \frac{2}{\sin 54^\circ}$$

$$= \frac{2 \times 4}{\sqrt{5} - 1} + \frac{2 \times 4}{\sqrt{5} + 1}$$

$$= \frac{8 \times 2\sqrt{5}}{4} = 4\sqrt{5}$$

$$\therefore pqr = 48\sqrt{5}$$